Using decomposition to address degeneracy in the dual simplex algorithm

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Algorithmic linear programming

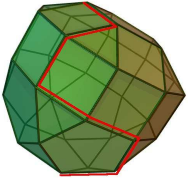
1947 Dantzig: primal **simplex algorithm**

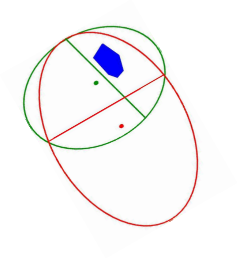
1954 Lemke and Beale: dual simplex algorithm

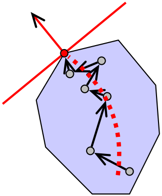
1979 Khachiyan: **ellipsoid method**

1984 Karmarkar: primal interior point/barrier algorithm

1989 Kojima et al: primal-dual **interior point algorithm**







Simplex method - efficient and effective

- Effective approach to solve linear programming problems.
- Fundamental for solving mixed integer programs.
- Algorithm developments have greatly improved efficiency.
  - Linear algebra,
  - picing and pivoting rules,
  - presolving techniques,
  - ...

Why improve the simplex method?

All about algorithm efficiency

- There are many features of the simplex methods that impact efficiency.
- One such feature is primal and dual degeneracy.

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- There are many features of the simplex methods that impact efficiency.
- One such feature is primal and dual degeneracy.

Goal: Reduce the degeneracy encountered in the simplex method to improve algorithm performance.

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Primal and dual degeneracy

Consider $P = \{\mathbf{x} \in \mathbb{R} | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and let $\mathbf{x}^*$ be a basic solution.

Primal Degeneracy

$\mathbf{x}^*$ is degenerate if more than $n - m$ elements are zero.

Dual Degeneracy

Degeneracy occurs if more than m primal variables have a reduced cost of zero.

Primal and dual degeneracy

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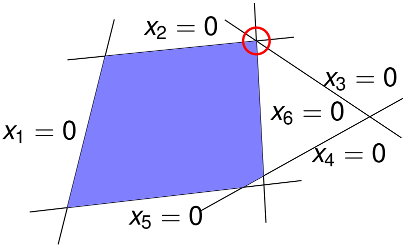

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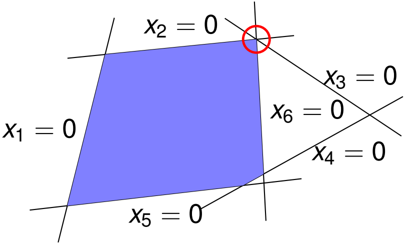
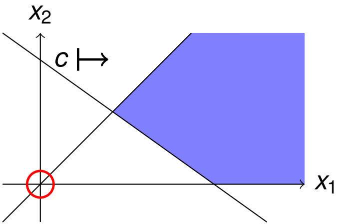
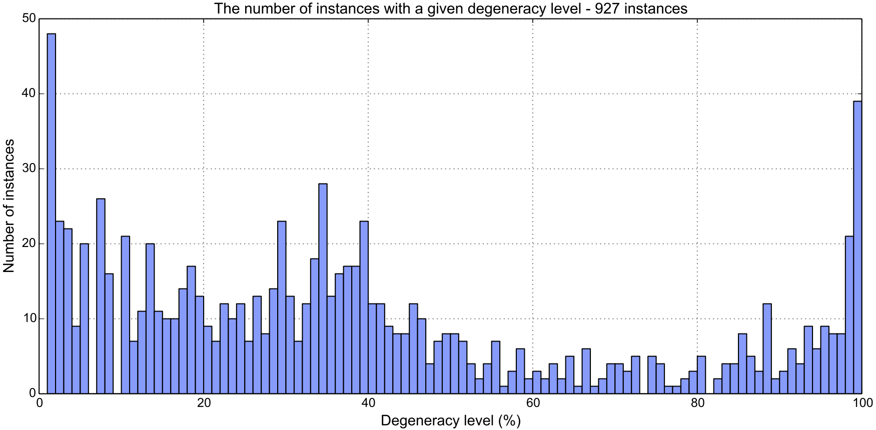




Prevalence of dual degeneracy



Collection of 1,226 LP instances - 299 have a degeneracy level less than 1%.

Approaches to address degeneracy

Primal degeneracy

Improved Primal Simplex - Elhallaoui et al. (2011)

- Developed with a focus on set partitioning problems.
- Decomposes problem to contain only non-degenerate primal variables.
- Further developments have resulted in an efficient solution approach.

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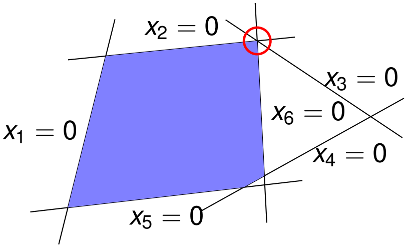

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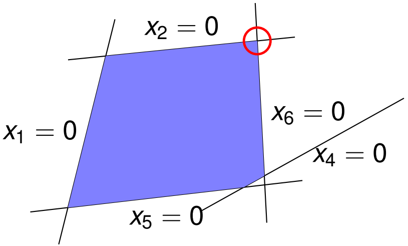

Approaches to address degeneracy

Dual degeneracy

- Perturbation
  - Modification of the objective function coefficients
- Bound flipping ratio test
  - Changing variable from one bound to another without affecting dual feasibility.
  - Larger steps are afforded.

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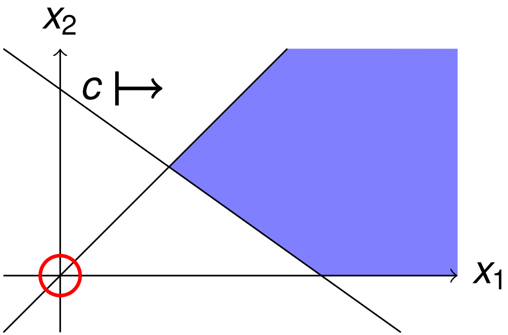

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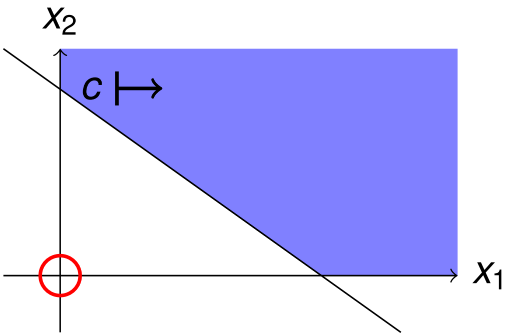

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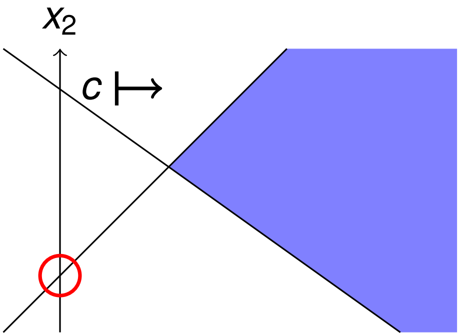

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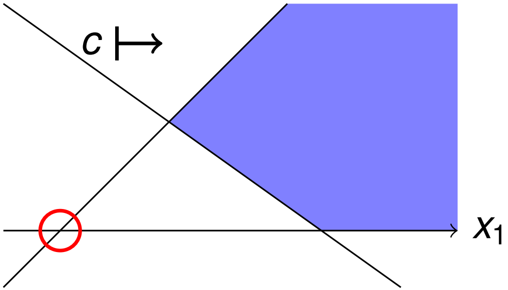

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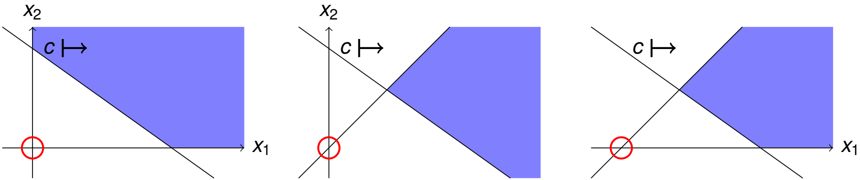
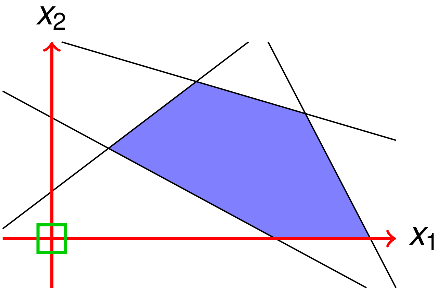


Row form basis matrix

Given $P = \max \left\{ c'x : \begin{pmatrix} -I \\ A \end{pmatrix} x \leq \begin{pmatrix} 0 \\ b \end{pmatrix} \right\}$.

- The row basis is an $n \times n$ matrix.
- Rows represent the constraints/variable bounds that are tight at the basic solution.



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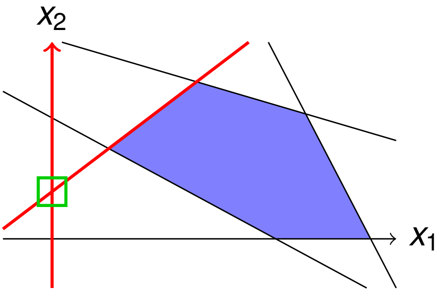

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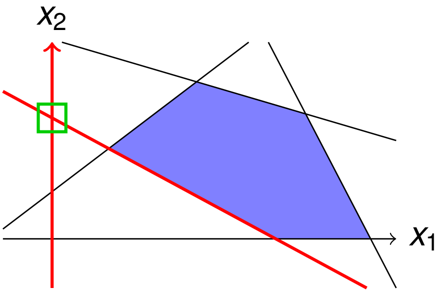

Formulation

Given $P = \max\{c'x : Ax \leq b\}$, and a row form basis B ($Q = B^{-1}$).

- Partition the indices of B by the dual multipliers.
  - $\mathcal{P}$ where $y_i > 0$ and $\mathcal{N}$ where $y_i = 0$.
- Reorder basis: $Q = (Q_{\cdot\mathcal{P}} | Q_{\cdot\mathcal{N}})$.
- Substitute $x = Q\bar{x}$, where $x = Q_{\cdot\mathcal{P}}\bar{x}_{\mathcal{P}} + Q_{\cdot\mathcal{N}}\bar{x}_{\mathcal{N}}$.
- Compatible rows identified by $A_{i\cdot}Q_{\cdot\mathcal{N}} = 0$, $i \in \{1, \ldots, m\}$.

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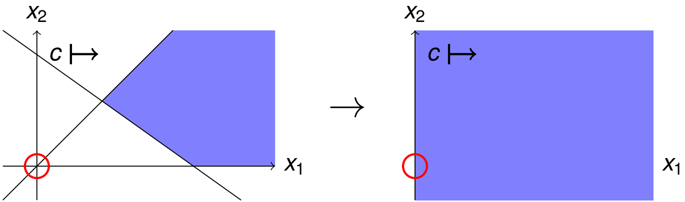
$$
\begin{aligned}
\max \quad & (c'Q_{\cdot\mathcal{P}})\bar{x}_{\mathcal{P}} + (c'Q_{\cdot\mathcal{N}})\bar{x}_{\mathcal{N}} \\
\text{subject to} \quad & A_{\mathcal{C}\cdot}Q_{\cdot\mathcal{P}}\bar{x}_{\mathcal{P}} + A_{\mathcal{C}\cdot}Q_{\cdot\mathcal{N}}\bar{x}_{\mathcal{N}} \leq b_{\mathcal{C}}, \\
& A_{\mathcal{I}\cdot}Q_{\cdot\mathcal{P}}\bar{x}_{\mathcal{P}} + A_{\mathcal{I}\cdot}Q_{\cdot\mathcal{N}}\bar{x}_{\mathcal{N}} \leq b_{\mathcal{I}}.
\end{aligned}
$$

Reduced problem



The reduced costs are: $c'Q = y_{\mathcal{B}}$.
Given $c'Q_{\cdot\mathcal{N}} = y_{\mathcal{N}} = 0$ and $A_{\mathcal{C}\cdot}Q_{\cdot\mathcal{N}} = 0$.

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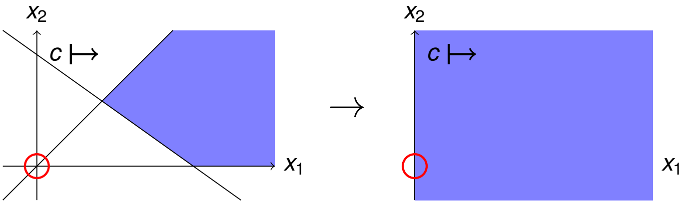

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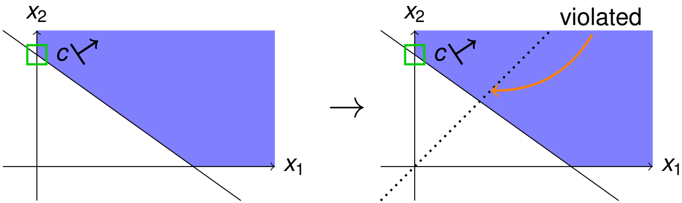
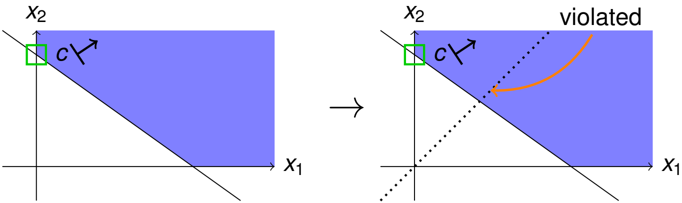
$$\begin{aligned} \max \quad & (c'Q_{\cdot\mathcal{P}})\bar{x}_{\mathcal{P}} \\ \text{subject to} \quad & A_{\mathcal{C}\cdot}Q_{\cdot\mathcal{P}}\bar{x}_{\mathcal{P}} \leq b_{\mathcal{C}}. \end{aligned}$$

Complementary problem



Know: Constraints currently at bounds.

Complementary problem



Know: Constraints currently at bounds.

$$\begin{aligned} \max \quad & s \\ \text{subject to} \quad & A_{i\cdot}x = b_i \quad \forall i \in \mathcal{P}^*, \\ & A_{i\cdot}x + s \leq b_i \quad \forall i \in \mathcal{I}^*, \\ & s \in \mathbb{R}. \end{aligned}$$

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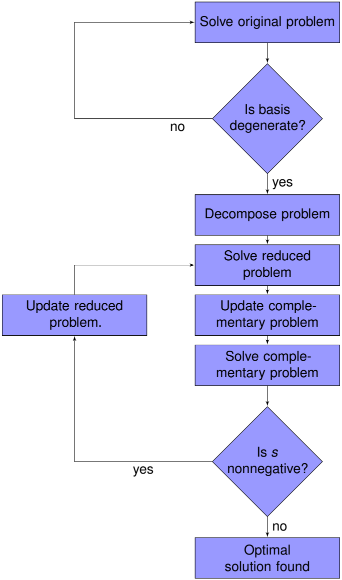
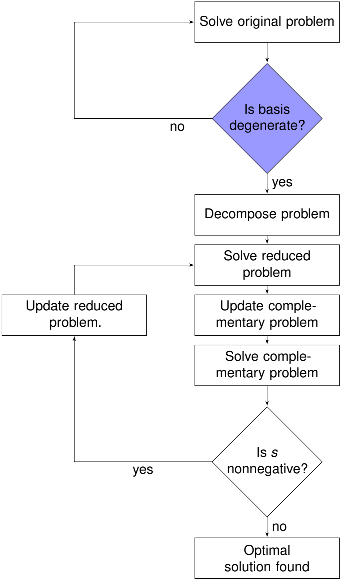


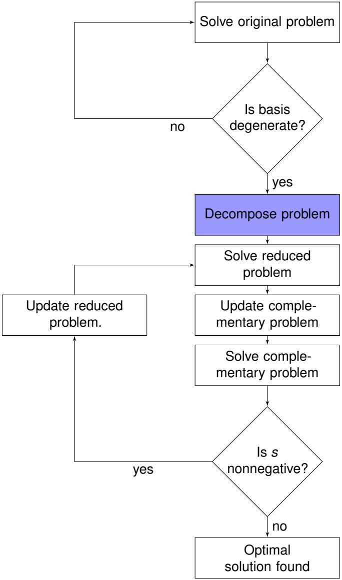
Solve original problem
Is basis degenerate?
no
yes
Decompose problem
Solve reduced problem
Update reduced problem.
Update complementary problem
Solve complementary problem
Is s nonnegative?
yes
no
Optimal solution found

Algorithm



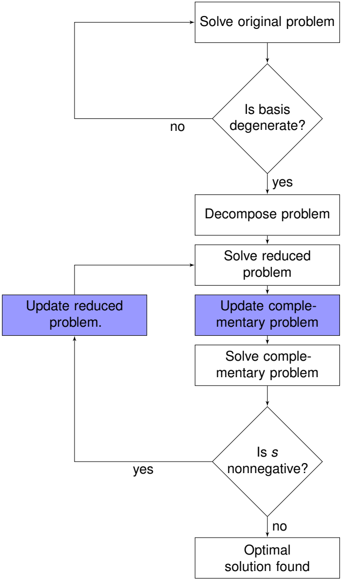
Must achieve an appropriate level of degeneracy.

Algorithm



Decomposition creates a reduced problem that eliminates current degeneracy

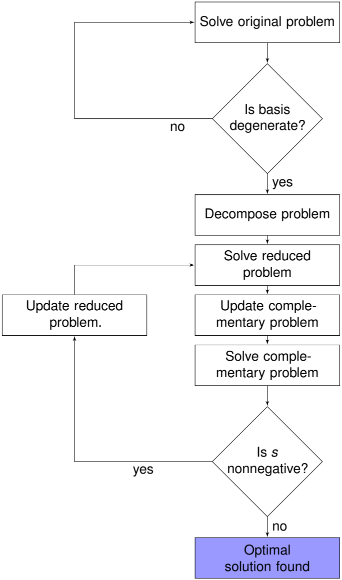
Algorithm



Complementary problem is updated with respect to the tight constraints.

Reduced problem is updated to include violated constraints.

Algorithm



Nonnegative s implies no violated constraints.

Implementation

Implemented within SoPlex
Simplex based LP solver developed at ZIB.

- Row representation available in SoPlex is necessary.
- Dual simplex used for original, reduced and complementary problems.
- No Phase 1 - degeneracy detected by subtracting bound shifts.

Complementary problem solved in the dual representation.

- Violated constraints are detected using a ratio test - comparing dual solutions from the reduced and complementary problems.

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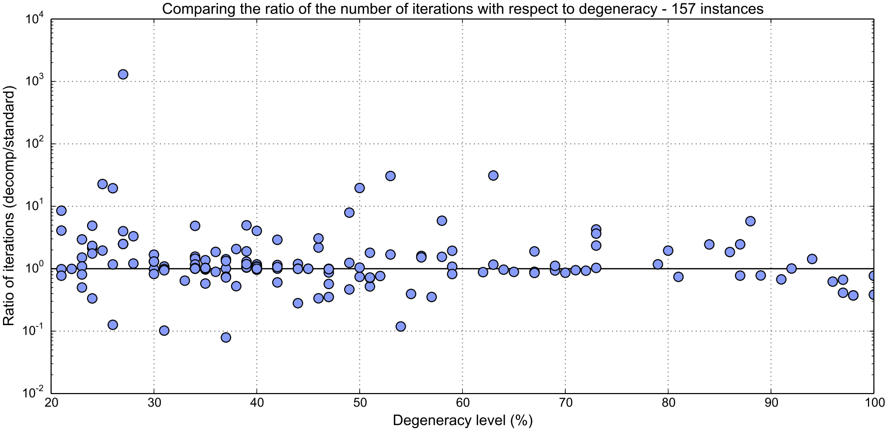
Test instances

- Set of instances collected from numerous LP and MIP benchmark testsets - total of 1,226
- Only considered those solved in one hour using dual algorithm of SoPlex in row representation and by the decomposition algorithm.
- Removed instances solved by SoPlex in less than 0.05 seconds and displayed average degeneracy less than 20%.
- Final testset contained 157 instances.

Iterations change



Conclusion

- Presented a decomposition approach designed to reduce the impact of degeneracy.
- Discussed the employed solution algorithm and the implementation in the freely available code of SoPlex.
- Demonstrated the potential of this approach to reduce the iterations of the dual simplex method.